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Energy levels of classical interacting fields in a finite domain in 1 + 1 dimensions

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Abstract. We study the behaviour of bound energy levels for the case of two classical interacting fields ϕ and χ in a finite domain (box) in 1+1 dimensions upon which we impose Dirichlet boundary conditions. The total Lagrangian contains a $\frac{\lambda}{4}\phi^4$ self-interaction and an interaction term given by $g\phi^2\chi^2$. We calculate its energy eigenfunctions and its corresponding eigenvalues and study their dependence on the size of the box (L) as well as on the free parameters of the Lagrangian: mass ratio $\beta = M_\chi^2/M_\phi^2$, and interaction coupling constants λ and g. We show that for some configurations of the above parameters, there exist critical sizes of the box for which instability points of the field χ appear.

1. Introduction

It is well known that quantum physical systems can significantly alter their behaviour when placed inside cavities. A modern paradigm is the famous Casimir effect [1] and more recently the so-called cavity quantum electrodynamics [2].

From a mathematical point of view, part of these studies can be translated into the general setting of differential equations for quantum fields, upon which are imposed suitable boundary conditions, in order to know their wavefunctions and energy eigenvalues. The above-mentioned subjects are essentially of a quantum nature. Nevertheless, it is well known that, for several applications, quantum fields can be thought of as classical fields upon which are added quantum corrections [3]. In this sense, although at a classical level, we can get a lot of information about the system under study.

In this paper we study the influence of boundary conditions on bound energy levels of a classical system of fields described by Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}M_\phi^2\phi^2 - \frac{\lambda}{4}\phi^4 + \frac{1}{2}(\partial_\mu\chi)^2 + \frac{1}{2}M_\chi^2\chi^2 - g\phi^2\chi^2 \qquad (1)$$

where λ, g are coupling constants.

Before we continue, we wish to make an important observation. In (1), the Lagrangian density of the field χ, i.e. $\mathcal{L} = \frac{1}{2}(\partial_\mu\chi)^2 + \frac{1}{2}M_\chi^2\chi^2$, does not have a state of least energy since its associated Hamiltonian is not positive definite. This, clearly, is due to the 'wrong' sign of the mass term. One way to solve this would be to add a term of self-interaction for the field χ, as usually happens in theories with spontaneous symmetry breaking. Of course we could also keep the original positive sign in the mass term. This will lead to different results from this work, but the techniques used are the same. In this work, we chose to keep the positive sign of the mass term and verify when the interaction term of the Lagrangian, given by $-g\phi^2\chi^2$, leads to a lower bound to the Hamiltonian density. We have found that the existence of a lower bound depends also on the boundary conditions. Therefore, the theory studied here must be understood as a toy model. (See the appendix for details.)

We consider only the simpler case of fields ϕ and χ inside a finite box (interval) in 1 + 1 dimensions. Of course, all discussion can be generalized to higher dimensions with a number of new differential equations and boundary conditions related to the geometry of the box. A quantum version of the theory, in a semiclassical approach, will be done elsewhere.

The equations of motion for the two fields are given by

$$-\partial^\mu \partial_\mu \chi + M_\chi^2 \chi - 2g\phi^2 \chi = 0 \tag{2}$$

$$-\partial^\mu \partial_\mu \phi + M_\phi^2 \phi - \lambda \phi^3 = 0. \tag{3}$$

In (3) we have neglected the term $2g\phi\chi^2$ which can be interpreted as the back-reaction of field χ on the mass term of ϕ. This can be achieved if we impose, for example, that $|\chi| \ll M_\phi/\sqrt{2g}$. Of course other regimes can be studied from (3) by adopting different approximations.

In a previous paper [4] we studied, in 1 + 1 dimensions, the case in which the field $\phi(x)$, unlike $\chi(x)$, is not influenced by finite boundary conditions of the box. So $\phi(x)$ is the Kink solution of (3) in $(-\infty, +\infty)$ [5]. In this case we can think of the χ field as placed in the presence of a fixed potential ϕ^2 and we showed that a level splitting appears (bifurcation point). This could be interpreted, in a semiclassical version of the theory, as χ-particle creation induced by squeezing the box below a critical size.

In this paper we take into account the same boundary conditions for both fields ϕ and χ. This means that the potential ϕ^2 depends on the size of the box. Therefore, we study the behaviour of the energy levels of the χ field by running the parameters L, β, λ, g, where $\beta = M_\chi^2/M_\phi^2$ is the mass ratio and L is the box size. Of course, the box size is an external parameter of the theory. We show below that classical instabilities appear for a critical size of box.

A family of static solutions of the classsical equation of motion to the field $\phi(x)$ are given by sn-type elliptic functions [6]

$$\phi_c(x) = \pm \frac{M_\phi\sqrt{2c}}{\sqrt{\lambda}\sqrt{1+\sqrt{1-2c}}} \mathrm{sn}\left(\frac{M_\phi x}{\sqrt{2}}\sqrt{1+\sqrt{1-2c}}, l\right) \tag{4}$$

where c is a parameter belonging to interval $(0, \frac{1}{2}]$ and

$$l = \frac{1}{-1 + \frac{1+\sqrt{1-2c}}{c}}. \tag{5}$$

Clearly, $l \in (0, 1]$. There also exists another family of solutions: see [6] for details. The above one was chosen because we can impose Dirichlet boundary conditions (DBCs) on their solutions. In general ϕ_c is a function of $x - x_0$, but we can put $x_0 = 0$ without loss of generality.

So, the equation of motion for the field χ can be written as

$$\left(-\partial^\mu\partial_\mu + M_\chi^2 - \frac{4gM_\phi^2 c}{\lambda(1+\sqrt{1-2c})}\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{2}}\sqrt{1+\sqrt{1-2c}}, l\right)\right)\chi = 0.$$

Since we are interested in stationary solutions we can write $\chi(x,t) = \mathrm{e}^{-\mathrm{i}\omega t}\psi(x)$, where ω are energy eigenvalues. With this, the previous equation can be written in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + \left(M_\chi^2 + \omega^2 - \frac{4gM_\phi^2 c}{\lambda(1+\sqrt{1-2c})}\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{2}}\sqrt{1+\sqrt{1-2c}}, l\right)\right)\psi(x) = 0. \tag{6}$$

In the next section we calculate the energy eigenvalues of (6) as well as study their dependence on the parameters of the theory, namely $g, \lambda, \beta = M_\chi^2/M_\phi^2, l$. For brevity we use $\omega \equiv \omega(\lambda, g, \beta; l)$, where the semicolon indicates that l is an external parameter of the theory. In section 5 we study the level shifts induced by changing the box size and interpret the results. In section 6 we conclude with some comments and list some topics for future work.

2. Lamé equation and boundary conditions

In this section we study the bound levels of two interacting fields which are confined inside a box (an interval in our case) of length L. We impose DBCs for both fields ϕ and χ (or ψ).

In [6] the authors showed that imposing DBCs on the field ϕ, given by (4), confined to a box of size L, must satisfy the condition

$$M_\phi L = 4\sqrt{1+l}K(l) \tag{7}$$

where $4K(l)$ is a period of the Jacobi elliptic functions $\mathrm{sn}(u, l)$ [7]. To get the above equation we impose DBCs at $x_1 = -\frac{L}{2}$ and $x_2 = \frac{L}{2}$. Observe that since the solutions (4) are continuous odd functions, the point $x_0 = 0$ is a root of all of them, and does not provide any information. So, it is sufficient to impose DBCs just, say, at $x_2 = \frac{L}{2}$. Moreover, since we take the same boundary conditions for both fields (the same kind of confinement), this implies that the same $l = l(L)$ obtained from the above equation must be substituted in the χ-field boundary conditions, in order to find its energy eigenvalues.

Of course, different boundary conditions can be imposed independently on the fields ϕ and χ. For example, in [4] we studied the extreme case where the box boundaries are transparent for the field ϕ, while χ satisfies DBCs.

We start by making the changes of variables,

$$\alpha = \frac{M_\phi x}{\sqrt{2}}\sqrt{1+\sqrt{1-2c}} \qquad \text{and} \qquad \omega^2 = \frac{(E-2)}{2}M_\phi^2$$

in equation (6) which can then be rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\psi(\alpha) = \left(\frac{8gc}{\lambda(1+\sqrt{1-2c})^2}\mathrm{sn}^2(\alpha, l) - 2\frac{(M_\chi^2 - M_\phi^2)}{M_\phi^2\sqrt{1+\sqrt{1-2c}}} - \frac{E}{(1+\sqrt{1-2c})}\right)\psi(\alpha). \tag{8}$$

On the other hand, from (5) we have

$$c = \frac{2l}{(l+1)^2}. \tag{9}$$

Thus (8) reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\psi(\alpha) = \left(4\frac{g}{\lambda}l\mathrm{sn}^2(\alpha, l) - \frac{(M_\chi^2 - M_\phi^2)}{M_\phi^2}(1+l) - \frac{E(1+l)}{2}\right)\psi(\alpha). \tag{10}$$

This differential equation has some important special properties. Since g and λ are positive we can write, without loss of generality,

$$4\frac{g}{\lambda} \equiv n(n+1) \tag{11}$$

where n is a positive real number.

So, the above equation can be rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\psi(\alpha) = \left(n(n+1)l\mathrm{sn}^2(\alpha, l) - (\beta-1)(1+l) - \frac{E(1+l)}{2}\right)\psi(\alpha) \tag{12}$$

where we have defined the adimensional mass ratio parameter as $\beta \equiv M_\chi^2/M_\phi^2$.

This is a generalized Lamé differential equation. In the literature the general form of this type of equation is given by [8]†

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\Lambda(\alpha) = (n(n+1)k\mathrm{sn}^2(\alpha, k) + C)\Lambda(\alpha) \tag{13}$$

where n is a positive real number, k is the parameter of the Jacobian elliptic function sn, and C is an arbitrary constant.

It is well known that the Lamé differential equation and more generally the Hill equation presents stability as well as instability bands in the plane of parameters (k, C) in the notation of equation (13). This stability is related to the spatial dependence of the solution. On the other hand, we are interested in stability for the time dependence. Using Floquet's theory the solution can be written as

$$\chi(x, t) = \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}rx}p(x)$$

where $p(x)$ is a periodic function. In our approach below (section 3) we show that even in the case for eigenvalues of the Lamé equation describing stable solutions for the spatial part ($r^2 > 0$), we can have unstable solutions in time, that is $\omega^2 < 0$.

Comparing (12) with (13), we obtain

$$C = -(\beta - 1)(1 + l) - \frac{E(1 + l)}{2} \qquad \text{and} \qquad k = l.$$

For the purpose we have in mind we only consider here the case where n is a positive integer. The case with n real, although more interesting, leads to eigenvalues which are difficult to calculate exactly and a full numerical treatment is necessary in order to obtain the eigenvalues we are interested in. For n an integer equation (12) or (13) is simply called the Lamé differential equation and can be solved analytically. So, our results can also be used as a test for the numerical solutions of more realistic cases which do not have exact solutions.

For n a positive integer the general solution of (12) is given by

$$\psi(\alpha) = AE_n^m(\alpha) + BF_n^m(\alpha)$$

where A and B are arbitrary constants and $E_n^m(\alpha)$ and $F_n^m(\alpha)$ are Lamé functions of the first and second kind, respectively [8]. The parameter m ranges from $\{-n, -n+1, \ldots, n-1, n\}$. Moreover, when n is a positive integer, if one of the solutions of the Lamé equation is a polynomial, then the second solution must be an infinite series. The polynomial solution is given by $E_n^m(\alpha)$ and the series solution by $F_n^m(\alpha)$ [8].

In this paper, we restrict our study to polynomial solutions. In other words, we search for solutions whose growth at infinity is of polynomial type. So our solutions are given only by

$$\psi(\alpha) = AE_n^m(\alpha). \tag{14}$$

Since we are interested only in eigenvalues, in the following we drop the arbitrary constant from the eigenfunctions (14).

Below we show the first eigenfunctions ($n = 1, 2, 3$) of (12) which are given by the Lamé functions. The case for n continuous will not considered in this work. Observe that the case $n = 0$, in principle, could be considered. But from (11), $n = 0$ implies $g = 0$ which in turn leads to a Hamiltonian which is not positive definite. Therefore the case $n = 0$ will be discarded. Of course, from (11) there is a minimal strength for the coupling constant, namely $g = \frac{\lambda}{2}$ (for $n = 1$). On the other hand, strong coupling ($n \to \infty$) leads to a new and more complicated Lamé functions. In this work we analyse only for the case small n: that is, the coupling constant has moderate strength. Yet even for these few cases, we will see that a rich phenomenology for the bound energy levels emerges.

† In our notation we take the parameter of the Jacobian elliptic function as k (with $k > 0$) instead of k^2 as in [8].

3. Eigenfunctions and eigenvalues of the ψ field

Using the results from [8], for $E_n^m(\alpha)$, as well as the form defined for the eigenvalues H, namely

$$H = \frac{1}{l}C$$

we list below the eigenfunctions $\psi(\alpha)$ (14) and their eigenvalues for (12).

Case I: $n = 1$ $(g = \frac{\lambda}{2})$

(1) $$\psi_1(x, l) = \mathrm{sn}\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) \qquad H_1^{-1}(l) = -1 - \frac{1}{l}. \tag{15}$$

(2) $$\psi_2(x, l) = \sqrt{\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - 1} \qquad H_1^{0}(l) = -\frac{1}{l}. \tag{16}$$

(3) $$\psi_3(x, l) = \sqrt{\frac{l\,\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - 1}{l}} \qquad H_1^{1}(l) = -1. \tag{17}$$

Case II: $n = 2$ $(g = \frac{3}{2}\lambda)$

(1) $$\psi_1(x, l) = \mathrm{sn}\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right)\sqrt{\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - 1}$$

$$H_2^{-1}(l) = -\frac{1}{l}(4 + l). \tag{18}$$

(2) $$\psi_2(x, l) = \mathrm{sn}\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right)\sqrt{\frac{l\,\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - 1}{l}}$$

$$H_2^{0}(l) = -\frac{1}{l}(4l + 1). \tag{19}$$

(3) $$\psi_3(x, l) = \sqrt{\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - 1}\sqrt{\frac{l\,\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - 1}{l}}$$

$$H_2^{1}(l) = -\frac{1}{l}(l + 1). \tag{20}$$

(4) $$\psi_4(x, l) = \mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - \frac{1}{1 + l + \sqrt{l^2 - l + 1}}$$

$$H_2^{2}(l) = -\frac{2}{l}(1 + l + \sqrt{l^2 - l + 1}). \tag{21}$$

(5) $$\psi_5(x, l) = \mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - \frac{1}{1 + l - \sqrt{l^2 - l + 1}}$$

$$H_2^{-2}(l) = -\frac{2}{l}(1 + l - \sqrt{l^2 - l + 1}). \tag{22}$$

Case III: $n = 3$ $(g = 3\lambda)$

(1) $$\psi_1(x,l) = \sqrt{\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - 1}\left(\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) + \frac{1}{-\sqrt{l^2-l+4}-l-2}\right)$$

$$H_3^{-2} = -\frac{2l+5}{l} - \frac{2}{l}\sqrt{l^2-l+4}. \qquad (23)$$

(2) $$\psi_2(x,l) = \sqrt{\frac{l\,\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - 1}{l}}\left(\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) + \frac{1}{-\sqrt{4l^2-l+1}-2l-1}\right)$$

$$H_3^{-1} = -\frac{5l+2}{l} - \frac{2}{l}\sqrt{4l^2-l+1}. \qquad (24)$$

(3) $$\psi_3(x,l) = \mathrm{sn}\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right)\sqrt{\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - 1}\sqrt{\frac{l\,\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - 1}{l}}$$

$$H_3^0(l) = -\frac{4}{l}(1+l). \qquad (25)$$

(4) $$\psi_4(x,l) = \mathrm{sn}\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right)\left(\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) + \frac{3}{\sqrt{4(l-1)^2+l}-2l-2}\right)$$

$$H_3^1 = -\frac{5}{l}(l+1) + \frac{2}{l}\sqrt{4(l-1)^2+l}. \qquad (26)$$

(5) $$\psi_5(x,l) = \sqrt{\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - 1}\left(\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) + \frac{1}{\sqrt{l^2-l+4}-l-2}\right)$$

$$H_3^2 = -\frac{2l+5}{l} + \frac{2}{l}\sqrt{l^2-l+4}. \qquad (27)$$

(6) $$\psi_6(x,l) = \sqrt{\frac{l\,\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) - 1}{l}}\left(\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) + \frac{1}{\sqrt{4l^2-l+1}-2l-1}\right)$$

$$H_3^3 = -\frac{5l+2}{l} + \frac{2}{l}\sqrt{4l^2-l+1}. \qquad (28)$$

(7) $$\psi_7(x,l) = \mathrm{sn}\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right)\left(\mathrm{sn}^2\left(\frac{M_\phi x}{\sqrt{1+l}}, l\right) + \frac{3}{-\sqrt{4(l-1)^2+l}-2l-2}\right)$$

$$H_3^{-3} = -\frac{5}{l}(l+1) - \frac{2}{l}\sqrt{4(l-1)^2+l}. \qquad (29)$$

In all cases we substituted with the help of (9), $\alpha = M_\phi x/\sqrt{1+l}$. For $l \in (0, 1]$ all the above eigenvalues H_n^m are negative.

4. Eigenvalues for DBCs

In this section we obtain the energy eigenvalues ω^2 by imposing DBCs at $x = \pm\frac{L}{2}$ on the solutions ψ_s above. Thus we obtain relations as $l \equiv l(L)$. We have made use of the relation $\omega^2 = \frac{(E-2)}{2}M_\phi^2$.

An important observation is now in order. In [6] a minimum value was determined for the relation $M_\phi L$, when $l \to 0$, namely, 2π. Using (7) (the same as (34) below), this result can be checked easily. Likewise, observe from (9) that if $l \to 0$ then $c \to 0$ and then by (4) we have that $\phi \to 0$. So, below the minimum value $M_\phi L = 2\pi$ the field ϕ vanishes. Therefore, in our calculation the only consistent eigenvalues are those that satisfy the condition $M_\phi L \geqslant 2\pi$, that is, those for which the field ϕ does not vanish. This argument will be used in several cases below. We now turn to the calculation of the energy eigenvalues for the eigenfunctions ψ_s.

Case I: $n = 1$ $(g = \frac{\lambda}{2})$

(1) $$\omega_1^2(\beta) = (1 - \beta) M_\phi^2. \tag{30}$$

Observe that in this case ω_1^2 does not depend on l.

(2) $$\omega_2^2(l, \beta) = \left(\frac{1}{1 + l} - \beta\right) M_\phi^2$$

where l satisfies

$$M_\phi L = 2\sqrt{1 + l}\, K(l).$$

Note that the $l = l(L)$ solution of this equation is not the same as that from (7). As previously mentioned, we are interested in obtaining energy eigenvalues of field χ with the same $l = l(L)$ used for field ϕ. Thus ω_2^2 must be discarded for the case $n = 1$.

(3) $$\omega_3^2(l, \beta) = \left(\frac{l}{1 + l} - \beta\right) M_\phi^2 \tag{31}$$

where l satisfies

$$\mathrm{sn}^2\left(\frac{M_\phi L}{2\sqrt{1 + l}}, l\right) = \frac{1}{l}. \tag{32}$$

Since $\mathrm{sn}^2(M_\phi L/2\sqrt{l + 1}, l) \leqslant 1$, l should satisfy $l \geqslant 1$. Since $l \in (0, 1]$, only $l = 1$ $(L = \infty)$ is a solution of (32).

Case II: $n = 2$ $(g = \frac{3}{2}\lambda)$

(1) $$\omega_1^2(l, \beta) = \left(\frac{4 + l}{1 + l} - \beta\right) M_\phi^2 \tag{33}$$

where l satisfies

$$M_\phi L = 4\sqrt{1 + l}\, K(l) \tag{34}$$

or

$$M_\phi L = 2\sqrt{1 + l}\, K(l). \tag{35}$$

Note that in this case only (34) satisfies the condition that the same $l = l(L)$ must be used for both fields ϕ and χ. According to this, solutions of equations like (35) must be discarded. So ω_1^2 is an allowed eigenvalue with l given by (34).

(2) $$\omega_2^2(l, \beta) = \left(\frac{4l + 1}{1 + l} - \beta\right) M_\phi^2 \tag{36}$$

where l satisfies (34) and also

$$\mathrm{sn}^2\left(\frac{M_\phi L}{2\sqrt{l + 1}}, l\right) = \frac{1}{l}. \tag{37}$$

Here, as in the case of (32), only $l = 1$ ($L = \infty$) is a solution. So ω_2^2 is also an allowed eigenvalue with l given by (34).

(3) $$\omega_3^2(\beta) = (1-\beta)M_\phi^2 \tag{38}$$

observe that ω_3^2 is independent of l and therefore of L.

(4) $$\omega_4^2(l,\beta) = \left(\frac{(2-\beta)(1+l) + 2\sqrt{l^2-l+1}}{1+l}\right) M_\phi^2 \tag{39}$$

where l satisfies

$$\mathrm{sn}^2\left(\frac{M_\phi L}{2\sqrt{l+1}}, l\right) = \frac{1}{1+l+\sqrt{l^2-l+1}}. \tag{40}$$

A numerical analysis shows that for $l \in (0, 1]$ we obtain, from the above equation, that the values of $M_\phi L$ belongs to the interval (1.57, 1.86). So, the eigenvalue ω_4^2 is not a consistent solution since for this case $M_\phi L < 2\pi$.

(5) $$\omega_5^2(l,\beta) = \left(\frac{(2-\beta)(1+l) - 2\sqrt{l^2-l+1}}{1+l}\right) M_\phi^2 \tag{41}$$

where l satisfies

$$\mathrm{sn}^2\left(\frac{M_\phi L}{2\sqrt{l+1}}, l\right) = \frac{1}{1+l-\sqrt{l^2-l+1}}. \tag{42}$$

This relation only has the solution $l = 1$ ($L = \infty$).

Case III: $n = 3$ ($g = 3\lambda$)

(1) $$\omega_1^2(l,\beta) = \left(\frac{2l+5+2\sqrt{l^2-l+4}}{1+l} - \beta\right) M_\phi^2. \tag{43}$$

Since we have a product in (23) we either get that l satisfies (35), which should be discarded, or that l satisfies

$$\mathrm{sn}^2\left(\frac{M_\phi L}{2\sqrt{l+1}}, l\right) = \frac{1}{l+2+\sqrt{l^2-l+4}}. \tag{44}$$

A numerical analysis shows that for $l \in (0, 1]$ $M_\phi L$ belongs to the interval (1.04, 1.36) and then $M_\phi L < 2\pi$. Therefore the eigenvalue ω_6^2 is not a consistent solution.

(2) $$\omega_2^2(l,\beta) = \left(\frac{5l+2+2\sqrt{4l^2-l+1}}{1+l} - \beta\right) M_\phi^2 \tag{45}$$

with l satisfying

$$\mathrm{sn}^2\left(\frac{M_\phi L}{2\sqrt{l+1}}, l\right) = \frac{1}{l} \qquad \text{or} \qquad \mathrm{sn}^2\left(\frac{M_\phi L}{2\sqrt{l+1}}, l\right) = \frac{1}{2l+1+\sqrt{4l^2-l+1}}. \tag{46}$$

Observe that the first equation is satisfied only for $l = 1$ ($L = \infty$), and for the second equation it is possible to show numerically that $M_\phi L$ belongs to the interval (1.36, 1.57). Thus this equation will not be considered since $M_\phi L < 2\pi$.

(3) $$\omega_3^2(l,\beta) = (4-\beta)M_\phi^2. \tag{47}$$

In this case ω_3^2 is independent of l and therefore of L.

(4) $$\omega_4^2(l,\beta) = \left(\frac{(5-\beta)(l+1) - 2\sqrt{4(l-1)^2+l}}{1+l}\right) M_\phi^2 \tag{48}$$

with l satisfying (34) or

$$\mathrm{sn}^2\left(\frac{M_\phi L}{2\sqrt{l+1}}, l\right) = \frac{3}{2l+2-\sqrt{4(l-1)^2+l}}. \tag{49}$$

Again, in this case only $l = 1$ ($L = \infty$) is a solution of (49). So ω_4^2 is an allowed eigenvalue with l satisfying (34).

(5)
$$\omega_5^2(l,\beta) = \left(\frac{2l+5-2\sqrt{l^2-l+4}}{1+l} - \beta\right) M_\phi^2 \tag{50}$$

with l satisfying an equation similar to (35) or

$$\mathrm{sn}^2\left(\frac{M_\phi L}{2\sqrt{l+1}}, l\right) = \frac{1}{l+2-\sqrt{l^2-l+4}}. \tag{51}$$

As in the previous case, only $l = 1$ ($L = \infty$) is a solution of (51).

(6)
$$\omega_6^2(l,\beta) = \left(\frac{5l+2-2\sqrt{4l^2-l+1}}{1+l} - \beta\right) M_\phi^2 \tag{52}$$

with l satisfying

$$\mathrm{sn}^2\left(\frac{M_\phi L}{2\sqrt{l+1}}, l\right) = \frac{1}{l} \qquad \text{or} \qquad \mathrm{sn}^2\left(\frac{M_\phi L}{2\sqrt{l+1}}, l\right) = \frac{1}{2l+1-\sqrt{4l^2-l+1}}. \tag{53}$$

These equations are also satisfied only for $l = 1$ ($L = \infty$).

(7)
$$\omega_7^2(l,\beta) = \left(\frac{(5-\beta)(l+1)+2\sqrt{4(l-1)^2+l}}{1+l}\right) M_\phi^2 \tag{54}$$

with l satisfying (34) or

$$\mathrm{sn}^2\left(\frac{M_\phi L}{2\sqrt{l+1}}, l\right) = \frac{3}{2l+2+\sqrt{4(l-1)^2+l}}. \tag{55}$$

By numerical analysis it is possible to show that $M_\phi L$ belongs to the interval $(2.04, 2.92)$. Thus these solutions must be discarded since $M_\phi L < 2\pi$. So ω_7^2 is an allowed eigenvalue with l satisfying (34).

The above study shows that only the eigenvalues ω_1^2 and ω_2^2 are allowed for $n = 2$, ω_4^2 and ω_7^2 for $n = 3$. Also there is a trivial one, namely ω_1^2 for $n = 1$, which coincides with ω_3^2 for $n = 2$.

In the next section, we study the behaviour of the energy eigenvalues ω^2, for a fixed β, running the external parameter of the theory $l \equiv l(L)$ continuously.

5. Level shifts induced by changing box size and points of instability

In this section we study the behaviour of level shifts with changing box size. The case $n = 1$, although possessing an allowed level, namely ω_1^2, does not depend on L in a non-trivial way. Therefore, we do not consider it interesting to our study.

For the cases $n = 2$ and 3 we have non-trivial results. In all cases below we fixed different values for the mass parameter β. Also, we demand classical stability for the eigenfunctions ψ_i $(i = 1, 2, 3)$. Classical stability means that the energy eigenvalues ω_i^2 are non-negative [3], so that the amplitude of field χ does not grow exponentially in time.

(A) Case $n = 2$ $(g = \frac{3}{2}\lambda)$. Considering only the energy eigenvalues $\omega_i^2 \geqslant 0$ in (33), (36) and (38) we obtain the following relations:

(a) $\beta \leqslant \frac{4+l}{1+l}$ for ω_1^2
(b) $\beta \leqslant \frac{4l+1}{1+l}$ for ω_2^2
(c) $\beta \leqslant 1$ for ω_3^2.

Since $l \in (0, 1]$, from these relations we get the allowed intervals for β. They are

(a) For ω_1, $\beta \in [\frac{5}{2}, 4)$.
(b) For ω_2, $\beta \in (1, \frac{5}{2}]$.
(c) For ω_3, $\beta \in [0, 1]$.

Below we study the behaviour of the energy eigenvalues ω_i^2 under the running of the external parameter L. In order to do this we fix some particular values of the mass ratio parameter β.

(1) $\beta = 0$ (figure 1).
Using (33), (36) and (38) we obtain
(a) $\omega_1^2 = (\frac{4+l}{1+l})M_\phi^2$
(b) $\omega_2^2 = (\frac{4l+1}{1+l})M_\phi^2$
(c) $\omega_3^2 = M_\phi^2$ satisfied for all L.
From figure 1 we can see that for a large box ($l = 1$ or $L = \infty$), ω_1 and ω_2 coincide at $\omega^2 = \frac{5}{2}M_\phi^2$.

(2) $\beta = 1$ (figure 2).
As before, from (33), (36) and (38) we obtain
(a) $\omega_1^2 = (\frac{3}{1+l})M_\phi^2$
(b) $\omega_2^2 = (\frac{3l}{1+l})M_\phi^2$
(c) $\omega_3^2 = 0$ satisfied for all L.
It is interesting to note that (see figure 2) for $L = \infty$, ω_1 and ω_2 converge to $\sqrt{\frac{3}{2}}M_\phi$, i.e. for a large box ($L = \infty$) the excited state of Dashen–Hasslacher–Neveu (DHN) [5] is obtained. Likewise, the ground state of the DHN model is also obtained, i.e. $\omega_3 = 0$. This can be proved directly from (6) by taking $l = 1$. Nevertheless, (6), or its equivalent (12), has an additional freedom by varying the parameter β.
Figure 2 shows that for $l \to 0$ $(M_\phi L \to 2\pi)$ [6], the energy eigenvalues ω_{2+} and ω_{2-} go to $\omega = 0$ for a critical size of the box, namely $L = \frac{2\pi}{M_\phi}$. This could suggest that this is an instability point of the field χ, induced by changing the external parameter L (box size). This is not the case here, because $l \to 0$ implies by (4) and (5) that $\phi = 0$ and we are left only with a free Lagrangian of the χ field and where no critical point exists.

(3) $\beta = 2$ (figure 3).
In this case, from (33), (36), (38) we have
(a) $\omega_1^2 = (\frac{2-l}{1+l})M_\phi^2$
(b) $\omega_2^2 = (\frac{2l-1}{1+l})M_\phi^2$
(c) $\omega_3^2 = -M_\phi^2$ satisfied for all L.
Observe that ω_3^2 is negative, so the classical configuration associated to this eigenvalue is unstable [3] and ω_{2+} turns out to be the new ground state. In the interval $l \in (0, \frac{1}{2})$ ω_2^2 is negative. Thus its classical configuration is unstable in this interval. Also observe that now the instability point occurs for a bigger size (not the minimal one) of the box. Here one can ask whether this kind of instability could lead to a 'condensate' in a quantum

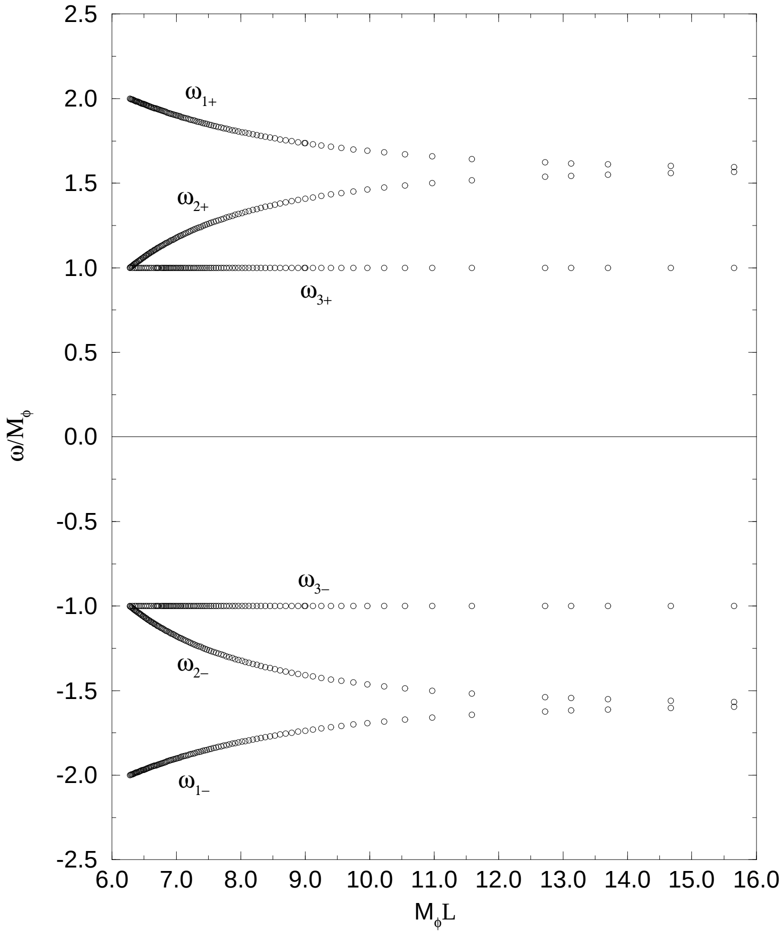

Figure 1. The energy eigenvalues for $\beta = 0$.

regime. A full proof of this, of course, requires a second quantization for the field χ, at least, in a semiclassical approach. Also, it is well known that inclusion of a nonlinear term for it in the total Lagrangian could lead to quantum condensates [9]. These lines are not pursued here, since our aim is only to show the behaviour energy levels for fields (in the classical limit) placed inside boxes, for a very simple geometry like an interval. For $l = 1$ ($M_\phi L = \infty$), ω_1 and ω_2 coincide at the value $\omega/M_\phi \sim 0.7$. (See figure 3.)

(4) $\beta = 3$ (figure 4).

In this case, using (33), (36) and (38) we have

(a) $\omega_1^2 = (\frac{1-2l}{1+l})M_\phi^2$

(b) $\omega_2^2 = (\frac{l-2}{1+l})M_\phi^2$

(c) $\omega_3^2 = -2M_\phi^2$ satisfied for all L.

As in previous cases, ω_3^2 is negative. Also, for $l \in (0, 1]$, $\omega_2^2 < 0$. So the classical configurations for ω_2 and ω_3 are unstable. Likewise in the interval $l \in (\frac{1}{2}, 1]$ we have that $\omega_1^2 < 0$, so its classical configuration is also unstable in this interval.

Figure 4 shows that by increasing the mass parameter β we get an instability point closer and closer to the critical size of the box $L = 2\pi/M_\phi$.

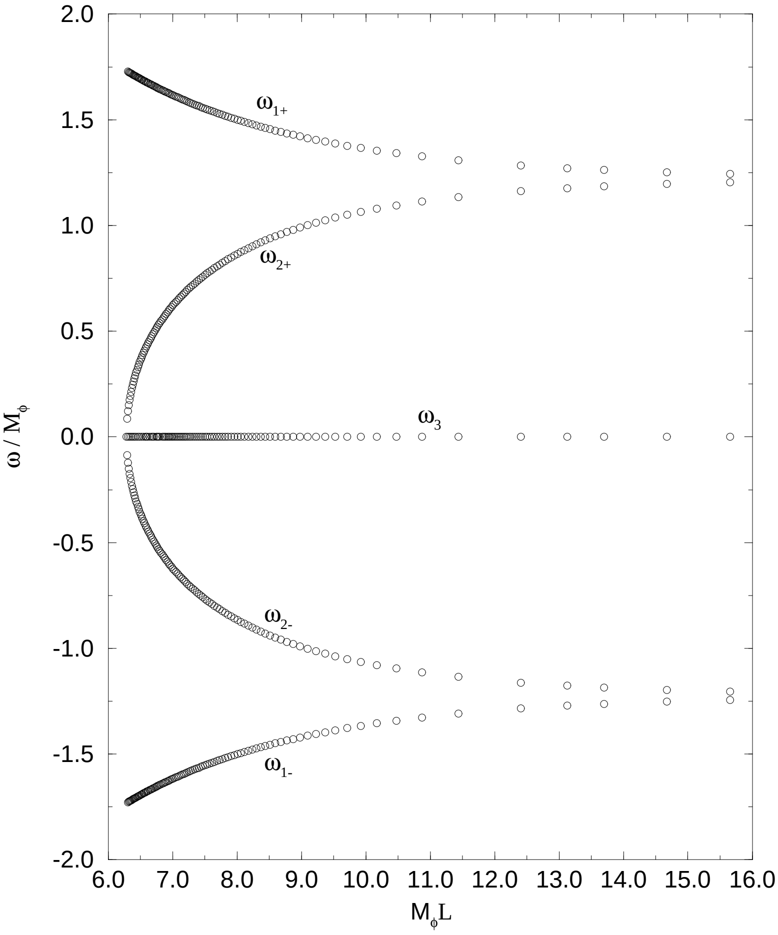

Figure 2. The energy eigenvalues for $\beta = 1$.

(5) $\beta = 4$.

Using (33), (36) and (38) we obtain

(a) $\omega_1^2 = -(\frac{3l}{1+l})M_\phi^2$

(b) $\omega_2^2 = -\frac{3}{1+l}M_\phi^2$

(c) $\omega_3^2 = -3M_\phi^2$.

We can see from these relations that all the energy eigenvalues are negative (for $l \neq 0$) and therefore their classical configurations will be unstable.

It is interesting to note that for $\beta \in [2, 3]$ the ω_2 disappears for $\beta \rightarrow \frac{5}{2}$ and for $\frac{5}{2} < \beta < 3$ only ω_1 survives and it is easy to see from figure 4 that its behaviour is the inverse of that of ω_2.

Therefore for $n = 2$, a changing of the box size induces the appearence of an instability point for the energy eigenvalues ω_1 or ω_2 for $\beta \in (1, 4)$.

(B) Case $n = 3$ *(*$g = 3\lambda$*).* For the case $n = 3$ the results are pretty much the same for its seven bound energy levels. However, in this case, only the eigenfunctions given by equations (26)

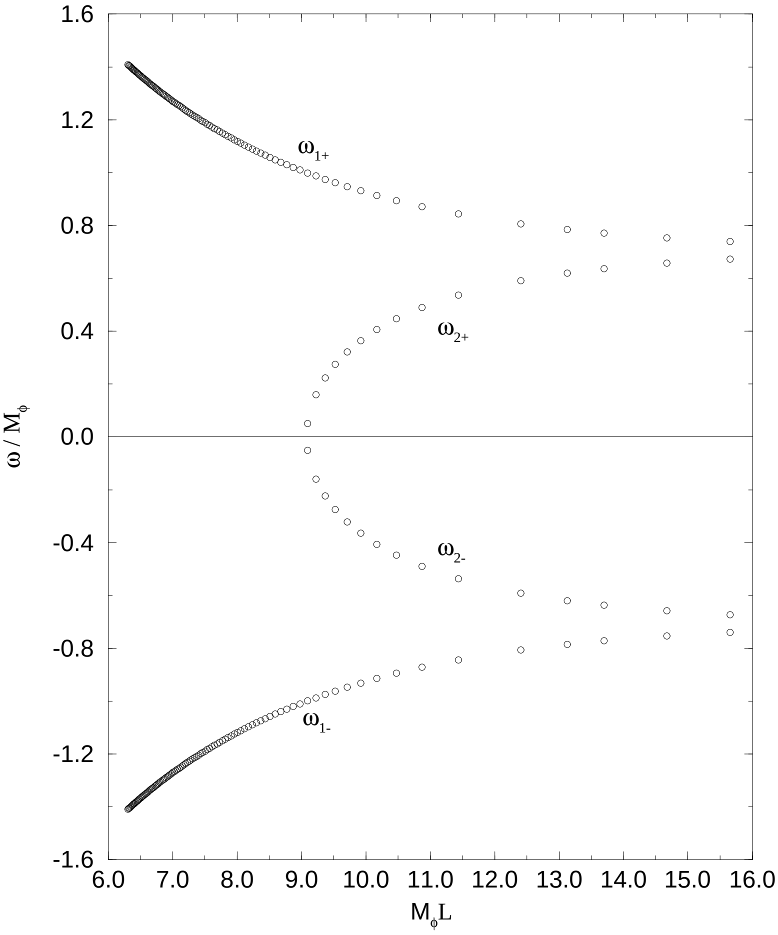

Figure 3. The energy eigenvalues for $\beta = 2$.

and (29) with respective energy eigenvalues given by (48) and (54) satisfy condition (7). The only great difference is that there is no instability point for the mass ratio parameter $\beta \in (4, 6)$. In fact, the instability points only exists for $\beta \in (1, 4]$ or $\beta \in [6, 9)$.

Another interesting point, as shown in [6], is that imposing periodic boundary conditions at $x = \pm\frac{L}{2}$ on the field ϕ, the same relation (7) is obtained and therefore all results from the Dirichlet case remain valid. Also, imposing periodic boundary conditions at $x = \pm\frac{L}{2}$ on the solutions ψ_s, it is possible to show that for the case $n = 2$, the eigenfunctions given by (19) and (20) lead to the same relation for $l \equiv l(L)$ given by (7). In the same way, for the case $n = 3$, the eigenfunctions given by (26) and (29) also lead to (7). Therefore all results of the Dirichlet case remain valid for periodic boundary conditions.

6. Conclusions

In this paper we have studied the energy eigenvalues ω^2 of a classical scalar field χ in 1 + 1 dimensions interacting with another classical scalar field ϕ through the Lagrangian $\mathcal{L}_{\text{int}} = g\phi^2\chi^2$, in a finite domain (box of size L). The energy eigenvalues depend on four

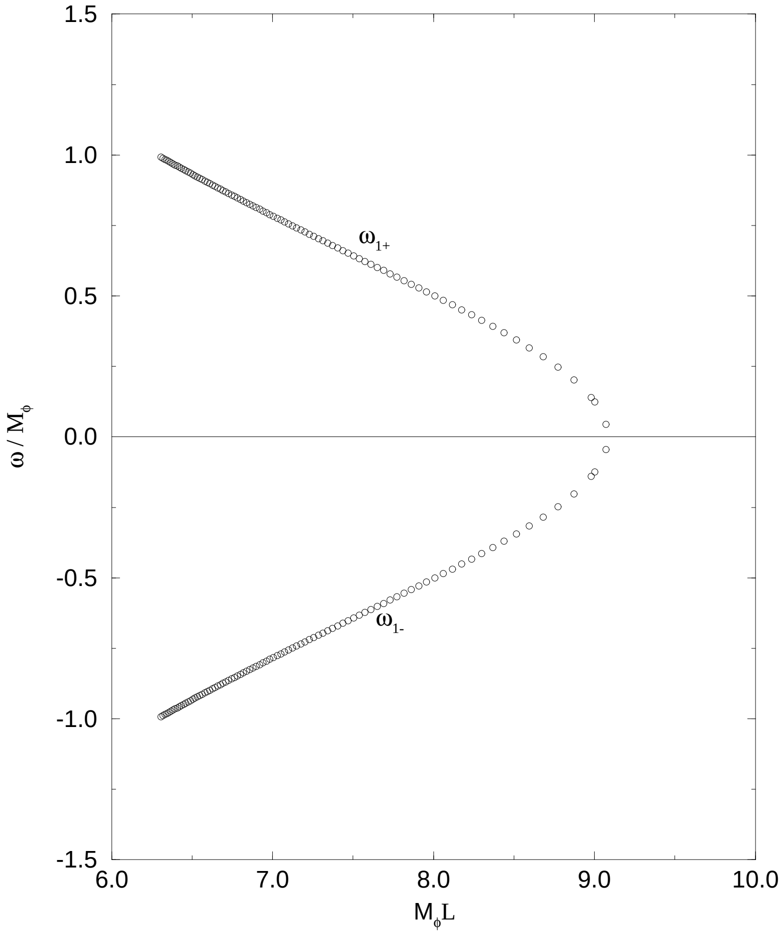

Figure 4. The energy eigenvalues for $\beta = 3$.

parameters: β (mass ratio parameter), coupling constants λ and g, and l (which is connected with the box size L). We fixed the coupling constant g by (11) for an arbitrary λ and we studied only the cases $n = 1, 2, 3$, which correspond to a moderate strength interaction constant g related to λ. For the more general case of n real a full numerical treatment is perhaps necessary. Next, we discussed the behaviour of the energy eigenvalues ω^2 by fixing the parameter β and changing the external parameter of the theory $l \equiv l(L)$, namely the size of the box.

For the case $n = 2$ ($g = \frac{\lambda}{2}$), we concluded that the instability points for the energy eigenvalues ω_1 or ω_2 occur for $\beta \in (1, 4)$. These instability points are obtained as a consequence of squeezing the box. Also, in figures 2–4 it is shown that increasing the mass ratio β gives ω_1 an instability point closer and closer to the minimal size of the box $L = \frac{2\pi}{M_\phi}$ while that of ω_2 is further and further from this value.

For the case $n = 3$ ($g = 3\lambda$), we have seven bound energy levels. Only two of them satisfy the DBC which in turn implies equation (7). Their behaviours are pretty much the same as $n = 2$, but instability points only exist for $\beta \in (1, 4]$ or $\beta \in [6, 9)$. For β in interval (4, 6) we get only stable solutions. For periodic boundary conditions all results obtained with DBCs remain unchangeable. Of course, other boundary conditions can be imposed leading to new behaviours of the energy levels under box squeezing.

Several interesting extensions and approaches can be made from this work. As stressed in the introduction, fields placed in cavities lead to new and sometimes unexpected behaviours of some systems. Although our approach is for classical fields it suggests that a quantization of the system studied above could lead to formation of a kind of 'condensate' just by squeezing the system in a box. This would require a semiclassical approach and will be done elsewhere. A more interesting case would be the inclusion of nonlinearities for the classical field χ, which leads to well known condensates for unbound domains [9], but now in finite domains. We think a full numerical treatment is also needed in this case.

Generalization of the above results to n spatial dimensions leads to more complex equations, as well as an enormous (infinite to be sure) variety of geometries for the shape of the box. Nevertheless, these kind of calculations for spherical symmetry could be interesting in order to study, for example, the bound state behaviour of matter fields in compact stars and in reheating theory and inflationary cosmology.

Acknowledgments

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Appendix. Boundary conditions and the positivity of the Hamiltonian

Consider the potential for the field χ, given by

$$V(\chi) = -\tfrac{1}{2}M_\chi^2\chi^2 + g\phi^2\chi^2$$

and we have

$$V'(\chi) = -M_\chi^2\chi + 2g\phi^2\chi = 0 \Rightarrow \chi = 0 \text{ (critical point).}$$

The second derivative of the potential $V(\chi)$ is given by

$$V''(\chi) = -M_\chi^2 + 2g\phi^2.$$

In order to make $V''(\chi) > 0$ we must impose that

$$\phi^2 > \frac{M_\chi^2}{2g}. \tag{56}$$

Thus the above condition should lead to the existence of a state of least energy (vacuum) of the field χ.

This is not the case for our example, because the condition (56) is not valid for field ϕ near the boundaries $x = \pm\frac{L}{2}$ since ϕ satisfies DBCs on them. This could lead to the total Hamiltonian density H not being positive definite. By writing H as

$$H = \frac{1}{2}(\dot{\chi}^2 + \dot{\phi}^2) + \frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\chi}{\mathrm{d}x}\right)^2 + \lambda\phi^4 + \frac{1}{2}M_\phi^2\phi^2 + \left(g\phi^2 - \frac{1}{2}M_\chi^2\right)\chi^2$$

we can easily see that the last term, between brackets, is not positive definite.

This work shows that unstable solutions appear for these kind of theories. This is because our example should be considered as a toy model in 1 + 1 dimensions. Nevertheless, it is possible that different boundary conditions could lead to unstable solutions without violating the positiveness of the Hamiltonian.

On the other hand, it is easy to see that if we change the sign of the mass term of the field χ in the Lagrangian, then we obtain only stable solutions. A detailed study of these more realistic solutions, including those for three spatial dimensions, will be done elsewhere using the method of investigation outlined in this paper.

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